EXERCISE – V

HINTS & SOLUTIONS

Sol.1 $y = x - x^2$; $y = mx$

$mx = x - x^2$

$x^2 = x(1 - m)$ or $x = 0,\ 1 - m$

$$\int (y_1 - y_2)\,dx = \int (x - x^2 - mx)\,dx$$

$$= \left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{1-m} = \frac{9}{2}$$

If $m < 1$ $(1 - m)^3 \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{9}{2}$

$(1 - m)^3 = 27 \Rightarrow m = -2$

If $m > 1$ then $1 - m$ will be negative

$$\left[(1-m)\frac{x^2}{2} - \frac{x^3}{3}\right]_{1-m}^{0} = \frac{9}{2}$$

$\Rightarrow (1 - m)^3 = -27$

$1 - m = -3 \Rightarrow m = 4$

Sol.2 $A = \int (y_1 - y_2)\,dx$

$$= \int_0^{1/\sqrt{2}} c\sqrt{1-x^2} - \left\{1 - \sqrt{1-x^2}\right\}dx$$

$$2A = \frac{2\sqrt{2}}{4} + \frac{\sqrt{2}\pi}{4} - \frac{2}{\sqrt{2}}$$

$$\text{required Area} = \pi - 2A = \pi - \frac{\pi}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

Sol.3 using the shifting of the origin parabola

$Y^2 = X$ ...(1)

where $y - 2 = y$ and $x - 1 = X$

so equation of tengent

$\Rightarrow X - 2Y + 1 = 0$ ...(2)

Area bounded by (1) and (2) and x-axis

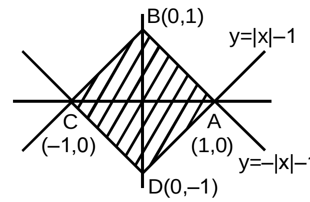
$$= 1/2(1) \times 1/2 + \int_0^1 \left(\frac{x+1}{2}\right) - \sqrt{x})dx$$

= 9 Sq units

Sol.4 $y = |x| - 1$

& $y = -|x| + 1$

required area = shaded region ABCD

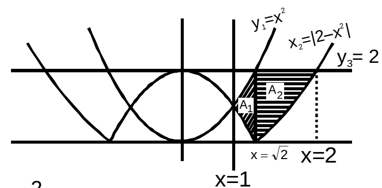


$$= 4 \times \left(\frac{1}{2} \times 1 \times 1\right)$$

$= 2$

Sol.5 Required bounded area $A = A_1 + A_2$

$$A_1 = \int_{x=1}^{x=\sqrt{2}} (y_1 - y_2)\,dx$$



$$= \int_1^{\sqrt{2}} (x^2 - (2 - x^2))\,dx$$

$$= \int_1^{\sqrt{2}} (2x^2 - 2)\,dx = 2\int_1^{\sqrt{2}} (x^2 - 1)\,dx$$

$$A_1 = \frac{4}{3} + \frac{2\sqrt{2}}{3}$$

$$A_2 = \int_{\sqrt{2}}^{2} (y_3 - y_2)\,dx = \int_{\sqrt{2}}^{2} (2 - (x^2 - 2))\,dx$$

$$= \int_{\sqrt{2}}^{2} [4 - x^2]\,dx = 8 - 4\sqrt{2} - \frac{8}{3} + \frac{2\sqrt{2}}{3}$$

$$= \frac{16}{3} - \frac{10\sqrt{2}}{3}$$

$A = A_1 + A_2$

$$= \frac{4}{3} + \frac{2\sqrt{2}}{3} + \frac{16}{3} - \frac{10\sqrt{2}}{3} = \frac{20}{3} - 4\sqrt{2} \text{ sq. units.}$$

Sol.6 Point of intersections of $y = ax^2$ & $x = ay^2$

are $(0, 0)$ & $\left(\frac{1}{a}, \frac{1}{a}\right)$

Hence $\int_0^{1/2} \left(\sqrt{\frac{x}{a}} - ax^2\right) dx = 1 \Rightarrow a = \frac{1}{\sqrt{3}}$ (as $a > 0$)

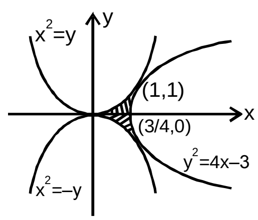
Sol.7 **(a)** $y = (x + 1)^2$ & $y = (x - 1)^2$ and the line $y = \frac{1}{4}$

the parabolas meet at (0, 1) and intersect the line $y = \frac{1}{4}$ at $x = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}$ and $\frac{3}{2}$

Hence the reqd. area $= 2\left[\int_0^{1/2} (x-1)^2\,dx\right] - \frac{1}{4}$

$$= \frac{2}{3}(x-1)^3\Big|_0^{1/2} - \frac{1}{4} = \frac{1}{3}$$

(b) The region bounded by the given curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$ is symmetrical about the x-axis. The parabolas $x^2 = y$ moreones the vertex of the curve $y^2 = 4x - 3$ is at $\left(\frac{3}{4}, 0\right)$



Hence the area of the region

$$= 2\left[\int_0^1 x^2\,dx - \int_{3/4}^1 \sqrt{4x-3}\,dx\right]$$

$$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \frac{1}{6}(4x-3)^{3/2}\Big|_{3/4}^1\right]$$

$$= 2\left[\frac{1}{3} - \frac{1}{6}\right] = \frac{1}{3} \text{ sq. units.}$$

(c) Let we have

$4a^2 f(-1) + 4a\, f(1) + f(2) = 3a^2 + 3a$ .....(1)

$4b^2 f(-1) + 4b\, f(1) + f(2) = 3b^2 + 3b$ .....(2)

$4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + 3c$ .....(3)

consider a quadratic equation

$4x^2 f(-1) + 4x\, f(1) + f(2) = 3x^2 + 3x$

or $[4f(-1) - 3]\,x^2 + [4f(1) - 3]x + f(2) = 0$ .....(4)

As equation (4) has three roots i.e. x = a, b, c. It is an identity.

$\Rightarrow f(-1) = \frac{3}{4}$; $f(1) = \frac{3}{4}$ and $f(2) = 0$

$$f(x) = \left(\frac{4-x^2}{4}\right) \qquad .....(5)$$

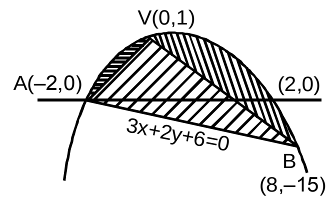
Let point A be (–2, 0) and B be $(2t, -t^2 + 1)$

Now as AB subtends a right angle at the vertex v(0, 1)

$\frac{1}{2} \times \left(-\frac{t^2}{2}\right) = -1$

$\Rightarrow$ t = 4

$\Rightarrow$ B(8, –15)



$$\text{Area} = \int_{-2}^{8}\left(\frac{4-x^2}{4} + \frac{3x+6}{2}\right)dx = \frac{125}{3} \text{ sq. units.}$$

Sol.8 **(i)** $I = \int_0^{\pi/2} (\sin x)^{\cos x}\,(\cos x.\cot x - \log(\sin x)^{\sin})\,dx$

$$= \int_0^{\pi/2} (\sin x)^{\cos x}\,dx = 1$$

(ii) Point of intersection of $-4y^2 = x$ and $x - 1 = -5y^2$ is (–4, –1) and (–4, 1)

Hence reqd. area

$$= 2\left|\int_0^1 (1-5y^2)\,dy - \int_0^1 (-4y^2)\,dy\right| = \frac{4}{3}$$

(iii) Point of intersection of $y = 3^{x-1}\log x$ and $y = x^x - 1$ is (1, 0)

Hence $\frac{dy}{dx} = \frac{3^{x-1}}{x} + 3^{x-1}\log 3\,.\log x$

$$\left.\frac{dy}{dx}\right|_{(1,0)} = 1$$

$y = x^x - 1 \Rightarrow \left.\frac{dy}{dx}\right|_{(1,0)} = 1$

If θ is the angle b/w the curves then $\tan\theta = 0$

$\Rightarrow \cos\theta = 1$

Sol.9 (a) I = $\int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}}-\sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}}-\sqrt{\frac{1+\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$= \int_0^{\pi/4}\frac{\left(1+\tan\frac{x}{2}\right)-\left(1-\tan\frac{x}{2}\right)}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

$$= \int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx = \int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$$

as $\tan\frac{x}{2} = t$

(b) (i) Differentiating the given equation, use get

$3y^2 y' - 3y' + 1 = 0$

$$y'(-10\sqrt{2}) = -\frac{1}{21}$$

Differentiation again use get

$6y(y')^2 + 3y^2 y'' - 3y'' = 0$

$$f''(-10\sqrt{2}) = -\frac{6.2\sqrt{2}}{(21)^4} = -\frac{4\sqrt{2}}{7^3\,3^2}$$

(ii) The reqd. area

$$= \int_a^b f(x)\,dx = x f(x)\Big|_a^b - \int_a^b x f'(x)\,dx$$

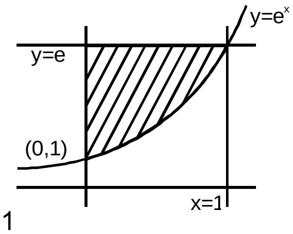
$$= b f(b) - a f(a) + \int_a^b \frac{x}{[f(x)^2-1]}dx$$

(iii) We have $y' = \frac{1}{3[1-[f(x)]^2]}$ which is even

Hence $\int_{-1}^{1} g'(x) = g(1) - g(-1) = 2g(1)$

Sol.10 Reqd. area

$$= \int_1^e \ell n\, y\, dy$$



$$= \left(y\,\ell n\,y - y\right)_1^e$$

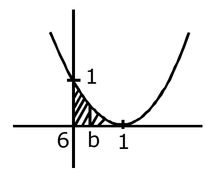
$= e - e - \{-1\} = 1$

Also $\int_1^e \ell n\,y\,dy = \int_1^e \ell n\,(e+1-y)\,dy$

Further the reqd. area = $(e \times 1) - \int_0^1 e^x\,dx$

Sol.11 $R_1 + R_2 = \int_0^1 (1-x)^2 . dx = 1/3$ ....(1)

$R_1 - R_2 = 1/4$ ....(2)



Now $2R_1 = 7/12$

$\Rightarrow R_1 = 7/24$

$\Rightarrow \int_0^b (1-x)^2\,dx = 7/24 \Rightarrow b = 1/2$

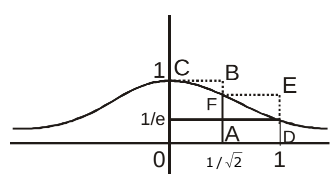
Sol.12 $R_1 = \frac{1}{2}\int_{-1}^{2} f(x)\,dx$;R_2

$$= \int_{-1}^{2} f(x)\,dx \Rightarrow R_2 = 2R_1$$

Sol.13 A,B,D

(B) $x \ge x^2$

$$\int_0^1 e^{-x}\,dx \le \int_0^1 e^{-x^2}\,dx$$



(D) S ≤ area OABC + area AFDE

$$s \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$$